# Problem 1

In the bouncing ball example above, find the height of the tenth rebound, and the distance traveled by the ball after it touches the ground the tenth time. Compare this distance with the total distance traveled.

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**Solution**

Consider a ball dropped initially from a height of 1 yard that bounces up two-thirds the fallen distance each time. The maximum height (in yards) after the $n$th bounce is then

$$1, \frac{2}{3}, \frac{4}{9}, \ldots, \left(\frac{2}{3}\right)^n.$$

Therefore, the maximum height after the tenth bounce is

$$\left(\frac{2}{3}\right)^{10} \approx 0.0173 \text{ yards.}$$

The total distance (in yards) that the ball travels by the $n$th bounce is

$$d(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + 2\displaystyle\sum_{i=1}^{n-1} \left(\frac{2}{3}\right)^i & \text{if } n \geq 2 \end{cases}.$$

In order to use the finite summation formula,

$$\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r},$$

make it so that the sum starts from $i = 0$.

$$d(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + \left[-2 + \displaystyle\sum_{i=0}^{n-1} 2\left(\frac{2}{3}\right)^i\right] & \text{if } n \geq 2 \end{cases}$$

Use the formula and simplify the result.

$$\begin{aligned} d(n) &= \begin{cases} 1 & \text{if } n = 1 \\ -1 + \dfrac{2\left[1 - \left(\frac{2}{3}\right)^n\right]}{1 - \left(\frac{2}{3}\right)} & \text{if } n \geq 2 \end{cases} \\ &= \begin{cases} 1 & \text{if } n = 1 \\ 5 - 6\left(\frac{2}{3}\right)^n & \text{if } n \geq 2 \end{cases} \\ &= 5 - 6\left(\frac{2}{3}\right)^n \end{aligned}$$

Therefore, the distance travelled by the tenth bounce is

$$d(10) = 5 - 6\left(\frac{2}{3}\right)^{10} \approx 4.90 \text{ yards.}$$

To find the total distance travelled, take the limit of $d(n)$ as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} d(n) = \lim_{n\to\infty}\left[5 - 6\left(\frac{2}{3}\right)^{n}\right] &= 5 - 6\lim_{n\to\infty}\left(\frac{2}{3}\right)^{n} \\
&= 5 - 6\lim_{n\to\infty}\left[e^{\ln\left(\frac{2}{3}\right)^{n}}\right] \\
&= 5 - 6\lim_{n\to\infty}\left[e^{n\ln\left(\frac{2}{3}\right)}\right] \\
&= 5 - 6\lim_{n\to\infty}\left[e^{-n\ln\left(\frac{3}{2}\right)}\right] \\
&= 5 - 6(0) \\
&= 5 \text{ yards}
\end{aligned}$$

The percentage of the total distance travelled by the tenth bounce is

$$\frac{d(10)}{d(\infty)} \times 100\% \approx 97.9\%.$$